16.5. Curl and divergence

Def (1) The del operator (or nabla operator) is

$$\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

(2) Given a vector field $\vec{F} = (P, Q, R)$, its curl and divergence are

$$\operatorname{curl}(\vec{F}) := \nabla \times \vec{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y) \leftarrow \text{a vector}$$

$$\operatorname{div}(\vec{F}) := \nabla \cdot \vec{F} = P_x + Q_y + R_z \leftarrow \text{a scalar.}$$

Note We will use the curl and divergence to study a different type of integrals in 16.8 and 16.9.

★ **Thm** A 3-dimensional vector field $\vec{F}$ is conservative on $\mathbb{R}^3$ if and only if $\operatorname{curl}(\vec{F}) = \vec{0}$.

Thm (Green's theorem, vector form)

Let $\vec{F}$ be a differentiable vector field on a domain D in $\mathbb{R}^2$. If the boundary ∂D is simple and positively oriented, then

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \underbrace{\operatorname{curl}(\vec{F}) \cdot \vec{k}}_{= Q_x - P_y} \, dA$$

Prop $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$ for any differentiable vector field $\vec{F}$.

Ex For each vector field on $\mathbb{R}^3$, find a potential function if it exists.

(1) $\vec{F}(x,y,z) = (3x^2y + z,\ x^3 + z^2,\ x + 2yz)$.

Sol $P = 3x^2y + z$, $Q = x^3 + z^2$, $R = x + 2yz$

$\text{curl}(\vec{F}) = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$

$= (2z - 2z,\ 1 - 1,\ 3x^2 - 3x^2) = (0,0,0)$

$\Rightarrow \vec{F}$ is conservative on $\mathbb{R}^3$.

For a potential function f, we want

$P = f_x$, $Q = f_y$, $R = f_z$

$\int P\,dx = \int 3x^2y + z\,dx = x^3y + xz$

$\int Q\,dy = \int x^3 + z^2\,dy = x^3y + yz^2$

$\int R\,dz = \int x + 2yz\,dz = xz + yz^2$.

$\Rightarrow$ A potential function is $f(x,y,z) = \boxed{x^3y + xz + yz^2}$

(2) $\vec{G}(x,y,z) = (ye^x,\ y^2e^z,\ zx)$

Sol $P = ye^x$, $Q = y^2e^z$, $R = zx$.

$\text{curl}(\vec{G}) = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$

$= (0 - y^2e^z,\ 0 - z,\ 0 - e^x) \neq (0,0,0)$

$\Rightarrow \vec{G}$ is not conservative on $\mathbb{R}^3$.

$\Rightarrow \vec{G}$ has $\boxed{\text{no potential functions}}$

☆ **Ex** The inverse square field is defined by

$$\vec{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$$

Compute $\operatorname{div}(\vec{F})$.

Sol $P = \frac{x}{(x^2+y^2+z^2)^{3/2}}$, $Q = \frac{y}{(x^2+y^2+z^2)^{3/2}}$, $R = \frac{z}{(x^2+y^2+z^2)^{3/2}}$

$$P_x = \frac{1 \cdot (x^2+y^2+z^2)^{3/2} - x \cdot \frac{3}{2}(x^2+y^2+z^2)^{1/2} \cdot 2x}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2-3x^2)}{(x^2+y^2+z^2)^3} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly, we get

$$Q_y = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{and} \quad R_z = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$$

(We get Q_y from P_x by swapping x and y
We get R_z from P_x by swapping x and z)

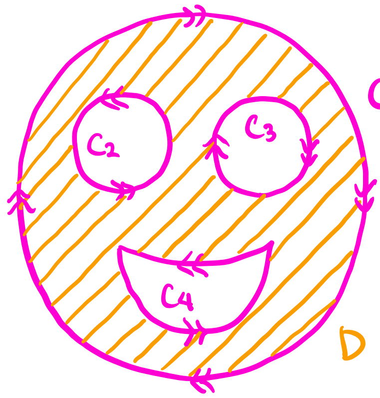
$$\operatorname{div}(\vec{F}) = P_x + Q_y + R_z$$

$$= \frac{(y^2+z^2-2x^2)+(x^2+z^2-2y^2)+(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{5/2}}$$

$$= \boxed{0}$$

Note The inverse square field is a model of many force fields.

Ex Consider the "smiley face" domain D given as follows:



Let $\vec{F}$ be a vector field with

$$\int_{C_1} \vec{F}\cdot d\vec{r} = 1, \quad \int_{C_2} \vec{F}\cdot d\vec{r} = 2, \quad \int_{C_3} \vec{F}\cdot d\vec{r} = 4, \quad \int_{C_4} \vec{F}\cdot d\vec{r} = 6.$$

Find $\iint_D \operatorname{curl}(\vec{F})\cdot\vec{k}\, dA$.

Sol C_1 : negatively oriented (outer, clockwise)

C_2 : negatively oriented (inner, counterclockwise)

C_3 : positively oriented (inner, clockwise)

C_4 : negatively oriented (inner, counterclockwise)

$$\Rightarrow \partial D = -C_1 - C_2 + C_3 - C_4.$$

$$\int_{\partial D} \vec{F}\cdot d\vec{r} = -\int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} - \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= -1 - 2 + 4 - 6 = -5$$

$$\iint_D \operatorname{curl}(\vec{F})\cdot\vec{k}\, dA = \int_{\partial D} \vec{F}\cdot d\vec{r} = \boxed{-5}$$

Green's thm